Optimal Aeroassisted Guidance Using Loh's Term Approximations

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This paper presents three guidance algorithms for aerocapture and/or aeroassisted orbital transfer with plane change. All three algorithms are based on the approximate solution of an optimal control problem at each guidance update. The chief assumption is that Loh's term may be modeled as a function of the independent variable only. The first two algorithms maximize exit speed for fixed exit altitude, flight-path angle, and heading angle. The third minimizes, in one sense, the control effort for fixed exit altitude, flight-path angle, heading angle, and speed. Results are presented that indicate the near optimality of the solutions generated by the first two algorithms. Results are also presented that indicate the performance of the third algorithm in a simulation with unmodeled atmospheric density disturbances.

Introduction

TWO interplanetary missions in the planning stages are the Mars Rover/Sample Return Mission and the Manned Mars Mission. Both missions require the delivery of heavy payloads into various Martian orbits. To reduce mass at liftoff or increase payload mass, aerocapture and aeroassisted orbital transfer are being considered for both missions. Thus, we are led to the problem of developing a hypersonic atmospheric guidance algorithm for such maneuvers. Ideally, one would like to be able to control all six exit states at a fixed exit time. Also, the algorithm must be sufficiently robust and/or adaptive to handle the possibly severe off-nominal conditions that may be encountered in the Martian atmosphere.

A number of studies of similar guidance algorithms have been undertaken.[1-10] This work extends some of the algorithms in Refs. 4–8. At each guidance update, an optimal control problem is solved, the problem being to maximize terminal speed while meeting end conditions on altitude, flight-path angle, and heading angle. One way in which the necessary conditions can be made integrable is to use a simplifying assumption on Loh's term. In Refs. 4–6, it is assumed dependent only on the independent variable. In Refs. 7 and 8, a zeroth-order solution is determined, and then higher-order terms are obtained through the Hamilton-Jacobi equation. Both techniques assume that Loh's term is near constant. However, over a typical trajectory, the term undergoes significant variation proportional to its magnitude due to altitude variation. Thus, these techniques are most appropriate when aerodynamic forces dominate, such as in the relatively dense atmosphere of the Earth. It is hoped that they will eventually be extended to flight through thinner atmospheres, such as at Mars.

In Refs. 4–6, the necessary conditions could only be integrated for the case where Loh's term was modeled as a two-segment, piecewise constant function of the independent variable. Here we expand from that point, retaining the common approximation that Loh's term may be modeled as a function of the independent variable only. First, we obtain an analytical integral of the necessary conditions for the multisegment, continuous, piecewise linear Loh's term approximation. Then a new set of variables is introduced. With these, simpler analytical integrals are obtained for a general Loh's term approximation as a function of the independent variable. Additionally, an iteration on two unknowns is eliminated. Finally, using these new variables, a variation of the guidance is introduced where the control effort is in some sense minimized, while meeting constraints on the exit altitude, flight-path angle, heading angle, and speed. By attempting to minimize the expected future guidance effort, the vehicle reserves the largest possible amount of control, which may be needed later to deal effectively with possible off-nominal conditions such as atmospheric variations. Necessary conditions for controlling the remaining exit states are derived, but no algorithm is presented for controlling those states.

Maximum Exit Speed Plane Change with Piecewise Linear Loh's Term

We start with the equations of motion for a vehicle in hypersonic atmospheric flight as given in Vinh et al.[11] Ignoring planet rotation terms, these are given by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = v \sin\gamma \tag{1a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{v \cos\gamma \cos\psi}{r \cos\phi} \tag{1b}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{v \cos\gamma \sin\psi}{r} \tag{1c}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{C_D S}{2m}\rho v^2 - g \sin\gamma \tag{1d}$$

$$v\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{C_L S}{2m}\rho v^2 \cos\eta - g \cos\gamma + \frac{v^2}{r}\cos\gamma \tag{1e}$$

$$v\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{C_L S}{2m}\rho v^2 \sin\eta - \frac{v^2}{r}\cos\gamma \cos\psi \tan\phi \tag{1f}$$

where

C_D = drag coefficient
C_L = lift coefficient
g = local gravitational acceleration
m/S = vehicle mass-to-surface area ratio

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r = vehicle distance from planet center
v = vehicle speed
γ = vehicle flight-path angle
η = bank angle
θ = vehicle longitude
ρ = atmospheric density at r
ϕ = vehicle latitude
ψ = vehicle heading angle

As in Refs. 4–6, we make the following approximations: we assume that γ and ϕ are small, that the gravitational term in dv/dt is negligible, and that only the lift term in $d\psi/dt$ is significant. We assume that the (angle-of-attack modulated) drag polar is parabolic so that we have

$$C_D = \frac{C_D^*}{2}(1+\lambda^2) \tag{2a}$$

$$C_L = \lambda C_L^* \tag{2b}$$

where C_L^* and C_D^* are the lift and drag coefficients at maximum L/D. Letting $\delta = \lambda \cos\eta$ and $\sigma = \lambda \sin\eta$, the equations of motion reduce to

$$\frac{dr}{dt} = v\gamma \tag{3a}$$

$$\frac{d\theta}{dt} = \frac{v \cos\psi}{r} \tag{3b}$$

$$\frac{d\phi}{dt} = \frac{v \sin\psi}{r} \tag{3c}$$

$$\frac{dv}{dt} = -\frac{C_D^* S}{4m}\rho v^2(1+\delta^2+\sigma^2) \tag{3d}$$

$$\frac{d\gamma}{dt} = \frac{C_L^* S}{2m}\rho v(\delta + M) \tag{3e}$$

$$\frac{d\psi}{dt} = \frac{C_L^* S}{2m}\rho v \sigma \tag{3f}$$

where M, Loh's term, is given by

$$M = \frac{2m}{C_L^* S}\left[\frac{v^2/r - g}{\rho v^2}\right]$$

For the algorithm under discussion in this section, we are concerned with maximizing exit speed for a given, nonzero, heading-angle change. The entry and exit altitude and flight-path angle are fixed, as well as the entry speed. We assume an exponential atmospheric density model $\rho = \rho_s \exp[(r_s - r)/\beta]$, where ρ_s is the atmospheric density multipler, r_s the atmospheric reference radius, and β the scale height. As in Refs. 4–6, we define the nondimensional density and speed by

$$w = \frac{C_L^* \rho \beta S}{2m}$$

$$\nu = 2\left(\frac{C_L^*}{C_D^*}\right) \ln(1/v)$$

Employing these variables, and considering only the equations relevant to our problem, Eqs. (3) are reduced to

$$\frac{d\gamma}{d\psi} = \frac{\delta + M}{\sigma} \tag{4a}$$

$$\frac{dw}{d\psi} = \frac{-\gamma}{\sigma} \tag{4b}$$

$$\frac{d\nu}{d\psi} = \sigma + \frac{1}{\sigma} + \frac{\delta^2}{\sigma} \tag{4c}$$

Let 0 and f subscripts denote initial and terminal values, respectively. Since maximizing exit speed is equivalent to minimizing ν_f, the constrained problem may be written as

minimize

$$F_0(\gamma, y) \equiv \int_0^{\psi_f} f_0(\dot\gamma, y)$$

subject to

$$\gamma(0) = \gamma_0$$

$$\gamma(\psi_f) = \gamma_f$$

$$F_2(\gamma, y) \equiv \int_0^{\psi_f} f_2(\gamma, y) = w_f - w_0$$

where

$$y = 1/\sigma$$

$$f_0 = y + (1/y) + (\dot\gamma^2/y) + yM^2 - 2\dot\gamma M$$

$$f_2 - y\gamma$$

and the dot notation indicates differentiation with respect to ψ.

We now proceed formally to obtain necessary conditions. For the variations of F_0 with respect to γ and y, one may take

$$(F_{0_\gamma}, \Delta\gamma) = \int (f_{0_\gamma}\Delta\gamma + f_{0_{\dot\gamma}}\Delta\dot\gamma)\, d\psi$$

$$(F_{0_y}, \Delta y) = \int f_{0_y}\Delta y\, d\psi$$

Also, employing Lagrange multipliers, one has the necessary conditions

$$F_{0_\gamma} + \lambda_2 F_{2_\gamma} = 0$$

$$F_{0_y} + \lambda_2 F_{2_y} = 0$$

Expanding the first of these, one has

$$\int (\lambda_2 f_{2_\gamma}\Delta\gamma + f_{0_{\dot\gamma}}\Delta\dot\gamma)\, d\psi = 0$$

Now we suppose $\dot\gamma \in C^2$ over $[0, \psi_f]$ except possibly at the points $\{\psi_i\}_{i=1}^n$, where $\dot M(\psi)$ has discontinuities. Then one may integrate by parts over the regions of continuity (see for instance Ref. 12) to obtain

$$\int_0^{\psi_1}\left(\lambda_2 f_{2_\gamma} - \frac{d}{d\psi}f_{0_{\dot\gamma}}\right)\Delta\gamma\, d\psi + f_{0_{\dot\gamma}}\Delta\gamma\Big|^{\psi_i^-}$$
$$+ \sum_{i=1}^{n-1}\int_{\psi_i}^{\psi_{i+1}}\left(\lambda_2 f_{2_\gamma} - \frac{d}{d\psi}f_{0_{\dot\gamma}}\right)\Delta\gamma\, d\psi + f_{0_{\dot\gamma}}\Delta\gamma\Big|_{\psi_i^+}^{\psi_{i+1}^-}$$
$$+ \int_{\psi_n}^{\psi_f}\left(\lambda_2 f_{2_\gamma} - \frac{d}{d\psi}f_{0_{\dot\gamma}}\right)\Delta\gamma\, d\psi + f_{0_{\dot\gamma}}\Delta\gamma\Big|_{\psi_n^+}^{\psi_f} = 0$$

Since this is true for all $\Delta\gamma$ in the set of admissible variations, one has

$$0 = -\frac{d}{d\psi}f_{0_{\dot\gamma}} + \lambda_2 f_{2_\gamma} = 2\dot M - \frac{2\ddot\gamma}{y} + \frac{2\dot\gamma}{y^2}\dot y$$
$$-\lambda_2 y \,\forall \psi \in (0,\psi_1) \cup (\psi_1,\psi_2) \cup \cdots \cup (\psi_n,\psi_f) \tag{5a}$$

and

$$f_{0_{\dot\gamma}}(\psi_i^-)=f_{0_{\dot\gamma}}(\psi_i^+) \qquad i=1,n \tag{5b}$$

Evaluating Eq. (5b), and noting that $\dot\gamma=(\delta+M)y$, one has

$$\delta(\psi_i^-)=\delta(\psi_i^+) \qquad i=1,n$$

Now, substituting into the second Lagrange multiplier equation leads to

$$1-(1/y^2)-(\dot\gamma^2/y^2)+M^2-\lambda_2\gamma=0 \tag{6}$$

which, noting that M and γ are continuous at ψ_i leads to

$$\sigma^2(\psi_i^-)=\sigma^2(\psi_i^+) \qquad i=1,n$$

From the preceding equations, we clearly have the continuity of δ, and the continuity, or at most a sign change, in σ at the points of discontinuity of $\dot M$. Although the possibility of a sign change in σ has not been fully explored, in the cases tested, continuity produced the minimum. The computational benefit of these simple corner conditions is that when using a multisegment piecewise linear M approximation, the terminal values of δ and σ at the end of one segment are simply the initial values at the start of the next.

Differentiating Eq. (6), and combining it with Eq. (5a) and the constraints of the problem, the set of necessary conditions becomes

$$2\dot M-\frac{2\ddot\gamma}{y}+\frac{2\dot\gamma}{y^2}\dot y-\lambda_2 y=0 \tag{7a}$$

$$\frac{2\dot y}{y^3}-\frac{2\dot\gamma\ddot\gamma}{y^2}+\frac{2\dot\gamma^2}{y^3}\dot y+2M\dot M-\lambda_2\dot\gamma=0 \tag{7b}$$

$$\left(1-\frac{1}{y^2}-\frac{\dot\gamma^2}{y^2}+M^2-\lambda_2\gamma\right)\bigg|_{\psi=0}=0 \tag{7c}$$

and

$$\gamma(0)=\gamma_0 \tag{8a}$$

$$\gamma(\psi_f)=\gamma_f \tag{8b}$$

$$\int_0^{\psi_f} f_2(\gamma,y)\,\mathrm{d}\psi=w_f-w_0 \tag{8c}$$

Equations (7a) and (7b) form the differential equations defining the control, and Eq. (7c) is a side condition from which the Lagrange multiplier may be obtained. Multiplying the first of the differential equations by $\dot\gamma/y$ and subtracting leads to

$$\dot y=y^2(\dot\gamma\dot M-yM\dot M) \tag{9a}$$

$$\ddot\gamma=\dot\gamma^2y\dot M-\dot\gamma y^2M\dot M-(\lambda_2/2)y^2+y\dot M \tag{9b}$$

Noting that $\delta+M=\dot\gamma/y$, and rearranging, leads to

$$\dot\delta=-(\lambda_2/2)y \tag{10a}$$

$$(\dot y/y^3)=\dot M\delta \tag{10b}$$

Combining these into a single, second-order equation, one obtains

$$\ddot\delta=(4\dot M/\lambda_2^2)\dot\delta^3\delta \tag{11}$$

Assuming linear $M=a+b\psi$, and integrating leads to

$$\frac{-1}{\dot\delta}=\frac{k}{2}\delta^2+c_1 \tag{12a}$$

$$-\psi+c_0=\frac{k}{6}\delta^3+c_1\delta \tag{12b}$$

where

$$k=\frac{4b}{\lambda_2^2}$$

so that δ is obtained as the solution to a cubic equation. Also, if $\delta(0)=\delta_0$ and $\dot\delta(0)=\dot\delta_0$,

$$c_1=-\left(\frac{1}{\dot\delta_0}+\frac{k}{2}\delta_0^2\right)$$

$$c_0=(k/6)\delta_0^3+c_1\delta_0 \tag{13}$$

Now, the integrals for γ, w, and ν will be obtained. For γ,

$$\Delta\gamma\equiv\int_0^{\psi_f}\dot\gamma=\int_0^{\psi_f}\frac{\delta+M}{\sigma}=\frac{-2}{\lambda_2}\left[\left(\frac{\delta^2}{2}+M\delta\right)\bigg|_0^{\psi_f}-b\int_0^{\psi_f}\delta\,\mathrm{d}\psi\right]$$

Changing variables to δ [and employing Eq. (12a)] in this last integral leads to

$$\Delta\gamma=\frac{-2}{\lambda_2}\left[\frac{bc_1+1}{2}\delta^2+M\delta+\frac{bk}{8}\delta^4\right]\bigg|_0^{\psi_f} \tag{14}$$

For w,

$$\begin{aligned}\Delta w&=\frac{2}{\lambda_2}\int_0^{\psi_f}\dot\delta(\gamma_0+\Delta\gamma)\,\mathrm{d}\psi\\&=\left\{\delta\gamma_0+\frac{2}{\lambda_2}\left[\frac{bc_1+1}{2}\delta_0^2+M\delta_0+\frac{bk}{8}\delta_0^4\right]\delta\right.\\&\quad\left.-\frac{2}{\lambda_2}\left[\frac{bc_1+1}{6}\delta^3+\frac{bk}{40}\delta^5\right]\right\}\bigg|_0^{\psi_f}-\frac{4}{\lambda_2}\int_0^{\psi_f}M\delta\dot\delta\,\mathrm{d}\psi\end{aligned}$$

But

$$\int_0^{\psi_f}M\delta\dot\delta\,\mathrm{d}\psi=\frac{M\delta^2}{2}\bigg|_0^{\psi_f}-\frac{b}{2}\int_0^{\psi_f}\delta^2\,\mathrm{d}\psi$$

where this latter integral may once again be evaluated by changing variables to δ and employing Eq. (12a) to lead to

$$\int_0^{\psi_f}\delta^2=-\left[\frac{k}{10}\delta^5+\frac{c_1}{3}\delta^3\right]\bigg|_0^{\psi_f}$$

so that

$$\begin{aligned}\Delta w=\frac{2}{\lambda_2}&\left\{\delta\gamma_0+\frac{2}{\lambda_2}\left[\frac{bc_1+1}{2}\delta_0^2+M\delta_0+\frac{bk}{8}\delta_0^4\right]\delta\right.\\&\left.-\frac{2}{\lambda_2}\left[\frac{M}{2}\delta^2+\frac{2bc_1+1}{6}\delta^3+\frac{3bk}{40}\delta^5\right]\right\}\bigg|_0^{\psi_f}\end{aligned} \tag{15}$$

Proceeding similarly for ν, one has

$$\Delta\nu=-\left\{\frac{2}{\lambda_2}\left[\delta+\frac{\delta^3}{3}\right]+\frac{\lambda_2}{2}\left[c_1^2\delta+\frac{kc_1}{3}\delta^3+\frac{k^2}{20}\delta^5\right]\right\}\bigg|_0^{\psi_f} \tag{16}$$

Thus, one has analytical expressions not only for the controls but also for the terminal constraints and objective functional as well.

For continuous, piecewise linear M, one must link together solutions over linear segments. The states at the end of one segment become the initial values for the following segment. That is, they are used in Eq. (13) to obtain c_0 and c_1, which are, in turn, used in Eqs. (12), (14), (15), and (16) for the following segment. Thus, for a three-segment, piecewise linear Loh's term approximation, one must sequentially solve three cubic equations.

Now, the original problem has end constraints on w and γ. To obtain the solution to this problem with piecewise linear M, one may use an iterative scheme on the two initial conditions of the control variables. That is, one chooses δ_0 and σ_0. Then Eq. (7c) is used to obtain λ_2. Equations (12), (14), and (15) may then be evaluated sequentially. The iteration is used to solve the nonlinear system of equations

$$\Delta\gamma(\delta_0,\sigma_0)=\gamma_f-\gamma_0$$

$$\Delta w(\delta_0,\sigma_0)=w_f-w_0$$

As in Refs. 4-6, a successive approximation algorithm is used to improve the piecewise linear approximation to Loh's term. An initial, estimated piecewise linear approximation is created. This is then used to generate a solution as already described. Then Eq. (16) is used sequentially to obtain the speed as a function of ψ. From this, one can obtain a new Loh's term as a function of ψ. A piecewise linear approximation is then fit to this by least squares, and the process is repeated. This successive approximation algorithm has consistently converged for the preceding equations. However, when the constraint on the terminal flight-path angle is removed, convergence problems often occur.

Maximum Exit Speed with General Loh's Term

In this section, we solve a slightly more general problem than that of the preceding section. In particular, we again maximize exit speed with constraints on exit altitude, flight-path angle, and heading angle. However, in this case, the heading-angle change may be zero. Again, Loh's term will be approximated as a function of the independent variable only, but in this case the functional form is unrestricted. Two benefits arise. First, no piecewise linear fit computations are necessary since the computed Loh's term is used directly. Second, the inner iteration of the previous algorithm to meet the end conditions on altitude and flight-path angle is no longer necessary. The constraints are met through the solution of a single quartic equation. These two benefits serve to significantly reduce the computational burden of the algorithm, and may also lead to more robust software.

Employing Eqs. (3), and defining $\nu=\ln v$ (a slight variation from the preceding section) and $u=\int_0^t(\rho v/\rho_s)\,\mathrm{d}t$, we have

$$\frac{\mathrm{d}\gamma}{\mathrm{d}u}=\frac{C_L^*S\rho_s}{2m}(\delta+M) \tag{17a}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}u}=-\frac{C_D^*S\rho_s}{4m}(1+\delta^2+\sigma^2) \tag{17b}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}u}=\frac{C_L^*S\rho_s}{2m}\sigma \tag{17c}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}u}=-\frac{\rho_s}{\beta}\gamma \tag{17d}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}u}=\frac{\rho_s\cos\psi}{\rho r_s} \tag{17e}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}u}=\frac{\rho_s\sin\psi}{\rho r_s} \tag{17f}$$

Note that with this transformation, the first three states (γ, ν, and ψ) depend only on δ, σ, and M. The next state ρ depends only on the first, and the last two (θ and ψ) depend only on the previous states. Thus, under the assumption that M is dependent only on the independent variable, a hierarchy is set up.

Viewing ρ and ψ as the controls rather than δ and σ, which are eliminated through

$$\delta=-2\frac{m\beta}{SC_L^*\rho_s^2}\rho''-M\equiv-(2k_2\rho''+M) \tag{18a}$$

$$\sigma=\frac{2m}{C_L^*S\rho_s}\psi' \tag{18b}$$

we obtain

$$\gamma=-\frac{\beta}{\rho_s}\rho'$$

$$\nu=\nu_0-k_1\int_0^u f_0\,\mathrm{d}u$$

$$=\nu_0-k_1\int_0^u\left[1+(2k_2\rho''+M)^2+k_3\psi'^2\right]\mathrm{d}u$$

$$\theta=\theta_0+\frac{\rho_s}{r_s}\int_0^u\frac{\cos\psi}{\rho}\,\mathrm{d}u$$

$$\phi=\phi_0+\frac{\rho_s}{r_s}\int_0^u\frac{\sin\psi}{\rho}\,\mathrm{d}u$$

where

$$k_1=\frac{C_D^*S\rho_s}{4m}$$

$$k_3=\frac{4m^2}{C_L^{*2}S^2\rho_s^2}$$

and prime notation indicates differentiation with respect to u. Once again letting 0 and f subscripts indicate initial and final conditions, respectively, our problem becomes

minimize

$$F_0=\int_0^U f_0\,\mathrm{d}u \tag{19a}$$

subject to

$$\rho(0)=\rho_0 \tag{19b}$$

$$\rho'(0)=-(\rho_s\gamma_0/\beta) \tag{19c}$$

$$\psi(0)=0 \tag{19d}$$

$$\rho(U)=\rho_f \tag{19e}$$

$$\rho'(U)=-(\rho_s\gamma_f/\beta) \tag{19f}$$

$$\psi(U)=\psi_f \tag{19g}$$

where U is the (unconstrained) value of the independent variable u at exit. We assume that M is dependent only on u and U (i.e., not ρ and ν). We assume dependence on U as well as u, since we often start the algorithm with large errors in U, but the exit value of M is relatively stable for the given exit conditions and near the optimal speed. Of course, ignoring the dependence on the dependent variables requires that the aerodynamic forces dominate.

Formally taking variations of F_0 we have

$$\delta F_0 = f_0\Big|_U \delta U + \int_0^U \left(f_{0_\rho}\delta\rho + f_{0_{\rho'}}\delta\rho' + f_{0_{\rho''}}\delta\rho'' + f_{0_\psi}\delta\psi + f_{0_{\psi'}}\delta\psi' + f_{0_U}\delta U\right) \mathrm{d}u$$

$$= f_0\Big|_U \delta U + \int_0^U \left(f_{0_{\rho''}}\delta\rho'' + f_{0_{\psi'}}\delta\psi' + f_{0_U}\delta U\right) \mathrm{d}u$$

For the first term in the integral, we have

$$\int_0^U f_{0_{\rho''}}\delta\rho''\,\mathrm{d}u = \int_0^U 4k_2(2k_2\rho'' + M)\delta\rho''\,\mathrm{d}u$$

$$= 4k_2\left\{(2k_2\rho'' + M)\delta\rho'\Big|_0^U - (2k_2\rho''' + M')\delta\rho\Big|_0^U + \int_0^U (2k_2\rho'''' + M''\delta\rho\,\mathrm{d}u\right\}$$

Noting that ρ and ρ' have fixed endpoints, one has

$$\delta\rho(0) = \delta\rho'(0) = 0$$

$$\delta\rho'(U) = -\rho''(U)\delta U$$

$$\delta\rho(U) = -\rho'(U)\delta U$$

which leads to

$$\int_0^U f_{0_{\rho''}}\delta\rho''\,\mathrm{d}u = 4k_2\left\{\left[\rho'(2k_2\rho''' + M') - \rho''(2k_2\rho'' + M)\right]\Big|_U \delta U + \int_0^U (2k_2\rho'''' + M'')\delta\rho\,\mathrm{d}u\right\}$$

Similarly,

$$\int_0^U f_{0_{\psi'}}'\delta\psi' = -2k_3\left\{\psi'^2\Big|_U \delta U + \int_0^U \psi''\delta\psi\,\mathrm{d}u\right\}$$

$$\int_0^U f_{0_U}\delta U\,\mathrm{d}u = 2\int_0^U (2k_2\rho'' + M)\frac{\mathrm{d}M}{\mathrm{d}U}\delta U\,\mathrm{d}u$$

Combining the preceding equations, we have

$$\delta F_0 = \left\{1 + (2k_2\rho'' + M)^2\Big|_U - k_3\psi'^2\Big|_U + 4k_2\left[\rho'(2k_2\rho''' + M') - \rho''(2k_2\rho'' + M)\right]\Big|_U + 2\int_0^U (2k_2\rho'' + M)\frac{\mathrm{d}M}{\mathrm{d}U}\mathrm{d}u\right\}\delta U$$
$$+ 4k_2\int_0^U (2k_2\rho'''' + M'')\delta\rho\,\mathrm{d}u + 2k_3\int_0^U \psi''\delta\psi\,\mathrm{d}u \tag{20}$$

Since this must be zero for all $\delta\rho$, $\delta\psi$, δU, we have the necessary conditions

$$0 = 2k_2\rho'''' + M''$$

$$0 = \psi''$$

$$0 = 1 + \left[(2k_2\rho'' + M)^2 - k_3\psi'^2 + 4k_2\rho'(2k_2\rho''' + M') - 4k_2\rho''(2k_2\rho'' + M)\right]\Big|_U + 2\int_0^U (2k_2\rho'' + M)\frac{\mathrm{d}M}{\mathrm{d}U}\mathrm{d}u \tag{21}$$

$$\rho(0) = \rho_0$$

$$\rho'(0) = -(\rho_s\gamma_0/\beta)$$

$$\psi(0) = 0$$

$$\rho(U) = \rho_f$$

$$\rho'(U) = -(\rho_s\gamma_f/\beta)$$

$$\psi(U) = \psi_f$$

Clearly, ψ is linear in u, so that

$$\psi = (\psi_f/U)u \tag{22}$$

Let $M(u,U) = \bar{M}(x)$ where $x = u/U$. Then, integrating the first of the necessary conditions, one has

$$\rho'' = 2a_2 + 6a_3u - (1/2k_2)\bar{M}(u/U) \tag{23}$$

with arbitrary constants a_2 and a_3. Letting $N(x) = \int_0^x \bar{M}(x)\,\mathrm{d}x$ and $O(x) = \int_0^x \bar{N}(x)\,\mathrm{d}x$, one has

$$\rho = a_0 + a_1u + a_2u^2 + a_3u^3 - (U^2/2k_2)O(u/U)$$

One can note at this point that, by combining Eqs. (18) and (23), one finds

$$\delta = 2k_2(2a_2 + 6a_3u)$$

Noting this and Eq. (22), we find the surprising result that the controls, σ and δ, are constant and linear in u, respectively, for any M of the form $M = \bar{M}(u/U)$.

Using the initial values of ρ and ρ', one has $a_0 = \rho_0$ and $a_1 = -\rho_s\gamma_0/\beta$. Then a_2 and a_3 are easly obtained from the exit conditions on ρ and ρ' by solving the two linear equations in two unknowns

$$\begin{pmatrix} U^2 & U^3 \\ 2U & 3U^2 \end{pmatrix}\begin{pmatrix} a_2 \\ a_3 \end{pmatrix} = \begin{bmatrix} \rho_f - a_0 - a_1U + (U^2/2k_2)O(1) \\ -(\rho_s\gamma_f/\beta) - a_1 + (U/2k_2)N(1) \end{bmatrix} \tag{24}$$

The only remaining unknown is U. This must be chosen to satisfy Eq. (21). It can be shown that with the aforementioned form for the a constants, Eq. (21) takes the form of a quartic, and thus an entirely analytical solution is possible with any such functional form for M. This analytical solution can be constrasted with Refs. 4–6 where two (or three) nonlinear equations in two (or three) unknowns must be solved (for constant or two-segment piecewise constant M), a significantly less robust process. In Ref. 8, a quartic equation is also obtained for the solution of this problem through a different method, although the choice of independent variable appears to be equivalent.

This solution assumed a known functional form for Loh's term, or in an actual computational system, a series of values of M at specified values of u. An initial estimate of M is used and an iteration is used to improve the approximation. The method used in this study was a successive approximation iteration, identical to that described in the preceding section, without the piecewise linear fit step of course. Although a successive approximation iteration is easily mechanized, a method with guaranteed convergence, such as an ellipsoid algorithm, may be preferable.

Optimal Four-State Guidance with General Loh's Term

In the last two sections, algorithms for maximizing exit speed for fixed exit altitude, flight-path angle, and heading angle are developed. However, for a planetary mission, one would often ideally prefer to control all six exit states at a fixed exit time. In particular, for aerocapture with small plane change, one may definitely not want to maximize exit speed. In this section, we minimize a cost function related to the amount of control used. Necessary conditions for minimizing this with constraints on all six exit states (but not exit time) are developed. These necessary conditions are solved for the reduced case where only altitude, flight-path angle, heading angle, and speed are controlled at exit.

We start with Eqs. (17), and choose

$$\int_0^U \lambda^2\, du \equiv \int_0^U (\delta^2+\sigma^2)\, du$$

as a cost function. In other words, we are approximately minimizing the integrated, squared angle of attack. This cost function was chosen not only for simplicity, but also to approximately produce the most unused control, which can then, instead, be used to deal with off-nominal conditions. In particular, if off-nominal conditions are encountered, one can then employ the unused angle of attack to correct flight-path angle, altitude, and heading angle. The speed can be indirectly controlled through changes in altitude (i.e., density). Thus, the optimization problem to be solved at each guidance update becomes

minimize

$$F_c=\int_0^U \left[(2k_2\rho''+M)^2+k_3\psi'^2\right] du \tag{25}$$

subject to

$$\rho(0)=\rho_0$$

$$\rho'(0)=-(\rho_s\gamma_0/\beta)$$

$$\psi(0)=0$$

$$\rho(U)=\rho_f$$

$$-\rho'(U)=(\rho_s\gamma_f/\beta)$$

$$\psi(U)=\psi_f$$

$$F_0=-k_1\int_0^U f_0\, du$$

$$\equiv -k_1\int_0^U \left[1+(2k_2\rho''+M)^2+k_3\psi'^2\right] du$$

$$=v_f-v_0$$

$$F_1=\frac{\rho_s}{r_s}\int_0^U \frac{\cos\psi}{\rho}\, du=\theta_f-\theta_0$$

$$F_2=\frac{\rho_s}{r_s}\int_0^U \frac{\sin\psi}{\rho}\, du=\phi_f-\phi_0$$

where $k1$, $k2$, and $k3$ are as defined earlier, and f and 0 subscripts indicate exit and initial (current) conditions, respectively. The endpoint U is free. Noting that $F_c=F_0-k_1U$, we see that this problem is equivalent to the maximum independent variable problem

maximize

$$F_m\equiv U \tag{26}$$

subject to the same constraints.

To derive simple necessary conditions, one can approximate M as a function of the independent variable only. This approximation is, of course, most defensible when atmospheric forces dominate, such as in the dense Earth atmospheric flight example of the next section.

Now, with (ρ,ψ,U) as controls, the variation of F_m is simply δU. The variation of F_0 was given in Eq. (20). The variations of F_1 and F_2 may also be obtained formally, leading to

$$\delta F_1=\frac{\rho_s}{r}\left\{\int_0^U \frac{-\cos\psi}{\rho^2}\,\delta\rho\, du+\int_0^U \frac{-\sin\psi}{\rho}\,\delta\psi\, du+\left(\frac{\cos\psi}{\rho}\right)\bigg|_U \delta U\right\}$$

$$\delta F_2=\frac{\rho_s}{r}\left\{\int_0^U \frac{-\sin\psi}{\rho^2}\,\delta\rho\, du+\int_0^U \frac{\cos\psi}{\rho}\,\delta\psi\, du+\left(\frac{\sin\psi}{\rho}\right)\bigg|_U \delta U\right\}$$

Through Lagrange multipliers, we obtain the Euler-Lagrange equations

$$\lambda_1 4k_1k_2(2k_2\rho''''+M'')-\lambda_2\frac{\cos\psi}{\rho^2}-\lambda_3\frac{\sin\psi}{\rho^2}=0 \tag{27a}$$

$$-\lambda_1 2k_1k_3\psi''-\lambda_2\frac{\sin\psi}{\rho}+\lambda_3\frac{\cos\psi}{\rho}=0 \tag{27b}$$

and the transversability condition

$$1+\lambda_1k_1\left[-4k_2^2\rho''^2+8k_2^2\rho'\rho'''+4k_2\rho'M'-k_3\psi'^2+1+M^2\right]+\lambda_2\frac{\cos\psi}{\rho}+\lambda_3\frac{\sin\psi}{\rho}\bigg|_U=0$$

One might note here that we have assumed $M=M(u)$ rather than $M=M(u/U)$ as in the preceding section. This additional complication has been unnecessary here because large variations in U are not experienced when using this algorithm as a guidance scheme.

Equations (27) are apparently not analytically solvable. It can be shown, as in Ref. 13 for instance, that Eqs. (27) possess a convergent power series solution in some sphere away from $\rho=0$.

Rather than attempting to solve Eqs. (27), for a first algorithm, we solve the simpler problem where the constraints on θ and ϕ are dropped. In this case, Eqs. (27) become

$$2k_2\rho''''+M''=0 \tag{28a}$$

$$\psi''=0 \tag{28b}$$

and the transversability constraint becomes

$$1+\lambda_1k_1\left[-4k_2^2\rho''^2+8k_2^2\rho'\rho'''+4k_2\rho'M'-k_3\psi'^2+1+M^2\right]\bigg|_U=0 \tag{29}$$

Equation (28b) clearly implies that ψ is linear, and combined with the constraints of Eq. (26) one has $\psi=(\psi_f/U)u$. To solve Eq. (28a), let M be any function of the form $M(u)$. Then

$$\rho=a_0+a_1u+a_2u^2+a_3u^3-\frac{1}{2k_2}\int_0^u\int_0^\zeta M(\xi)\, d\xi\, d\zeta \tag{30}$$

where in the actual system the integration is performed numerically. Once again, one has the surprising result that the controls, σ and δ, are constant and linear in u, respectively.

As in the preceding section, a_0, a_1, a_2, and a_3 are easily obtained from the constraints on $\rho(0)$, $\rho'(0)$, $\rho(U)$, and $\rho'(U)$. Condition (29) is clearly moot as long as the term multiplying

λ_1 is nonzero. All that remains is to find the U that satisfies $F_0 = v_f - v_0$. In the cases run so far, this has generally produced two solutions. Because we are maximizing U, the larger is chosen.

As in the preceding section, this solution assumed a known approximation to M, or in actuality, a set of values of M at specified values of u. The same comments regarding the use of iteration to improve this approximation apply here.

Results

We will now examine results from all three of the algorithms presented earlier. We will first examine the results of the two maximum exit speed algorithms. In particular, we will look at the optimal controls and resulting trajectories that each produces given a set of endpoint constraints. We will not examine their performance as guidance algorithms in the presence of unmodeled effects. However, for the third algorithm, we will examine the results that are obtained when using it as a guidance algorithm in the presence of rather severe unmodeled atmospheric density variations. In all cases, numerical simulation has been employed to verify the results.

The following nominal planet and vehicle parameters were used in all cases. These parameters approximate the realistic models of Ref. 14. They are

$$\rho_s = 2.667 \times 10^5 \text{ kg/km}^3, \qquad r_s = 6439.0 \text{ km}$$

$$\beta = 6.66 \text{ km}, \qquad g = \mu/r^2, \qquad \mu = 398603.2 \text{ km}^3/\text{s}^2$$

$$C_L^* = 0.16, \quad C_D^* = 0.16/2.4, \quad m/S = 1.946 \times 10^8 \text{ kg/km}^2$$

Additionally, the initial conditions for all cases were

$$r_0 = 6439.0 \text{ km}, \qquad \theta_0 = 0 \text{ deg}, \qquad \phi_0 = 0 \text{ deg}$$

$$v_0 = 7.83 \text{ km/s}, \qquad \gamma_0 = -1.57 \text{ deg}, \qquad \psi_0 = 0 \text{ deg}$$

For the two maximum speed algorithms, the end conditions were $r_f = r_0$, $\gamma_f = 0$ deg, and $\psi_f = 40$ deg.

The first algorithm (plane change with piecewise linear Loh's term) was run with a three-segment approximation. The slope discontinuity positions were fixed at 0.45 and 0.55 rad. The solution was found to be $\delta_0 = 0.0009$, $\sigma_0 = 0.9837$, and $\lambda_2 = -1.1795$. The terminal speed was 5.818 km/s. Figure 1 depicts the final Loh's term fit. Here, the piecewise linear fit was performed by matching the initial value of Loh's term, and then matching, in a least-squares sense, the derivative of Loh's term with respect to ψ throughout the trajectory. It should also be noted that, since the control was calculated only once at the initial time, the control was modified to be $\delta \equiv \delta + \tilde{M} - M$ where $\tilde{M} - M$ is the difference between the piecewise linear approximation and the actual values of Loh's term. With this modification, the end conditions are met almost exactly in the numerical simulation. This may result in a slightly inferior objective function value than what would be achieved through use of the algorithm as a guidance scheme. Figure 2 depicts the control history as a function of heading angle. There is a clear discontinuity in the derivative only at one of the two points where the Loh's term approximation segments meet.

Reducing the piecewise linear approximation to a two-segment fit with discontinuity at 0.5 rad did not significantly degrade the results. The solution was $\delta_0 = -0.0077$, $\sigma_0 = 0.9805$, and $\lambda_2 = -1.3988$, resulting in a terminal speed of

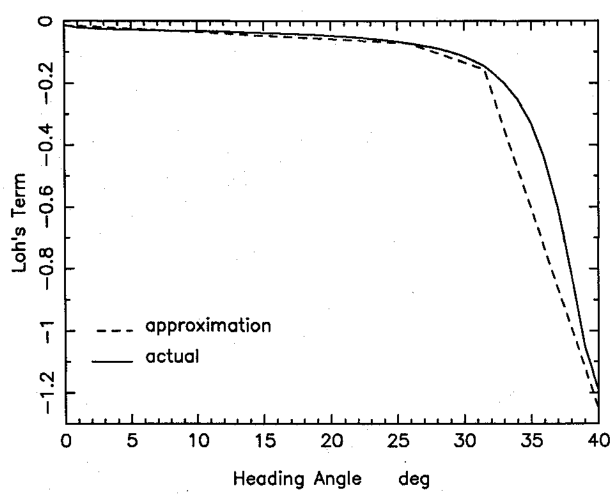

Fig. 1 Loh's term (actual and three-segment approximation).

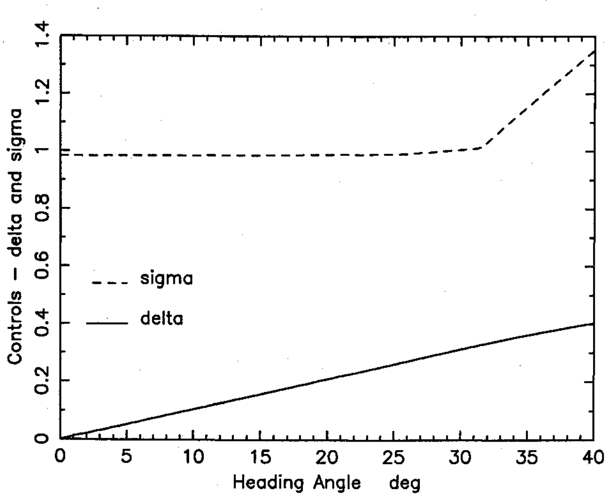

Fig. 2 Control histories for three-segment approximation.

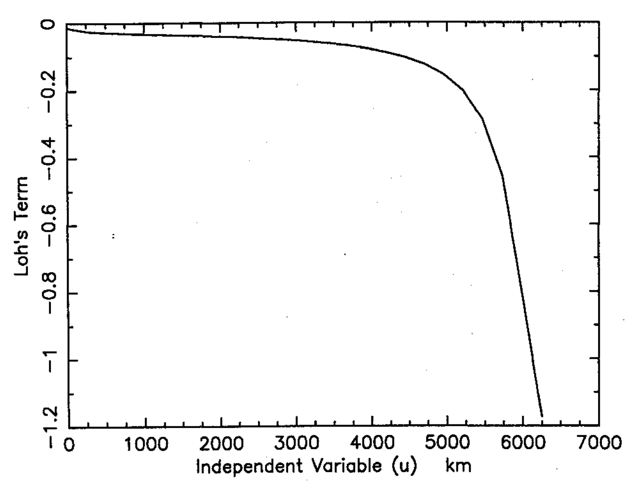

Fig. 3 Loh's term (25-point approximation).

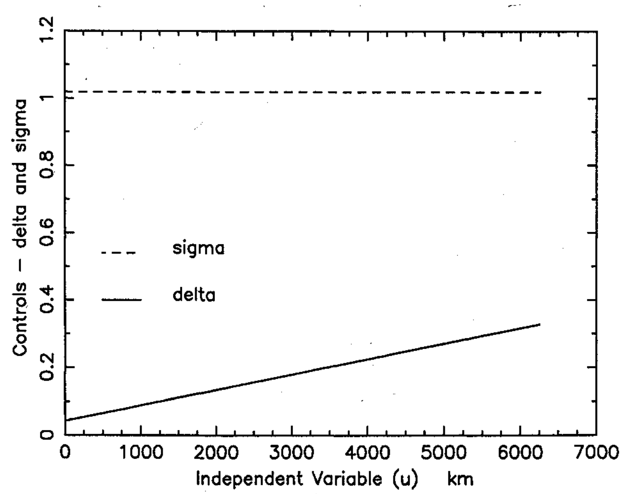

Fig. 4 Control histories for 25-point approximation.

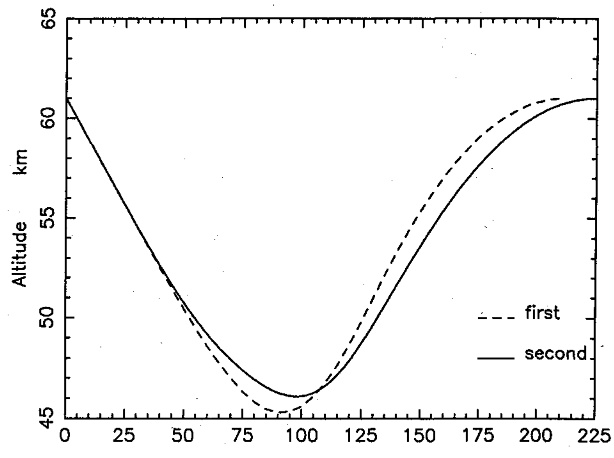

Fig. 5 Altitude profile comparison.

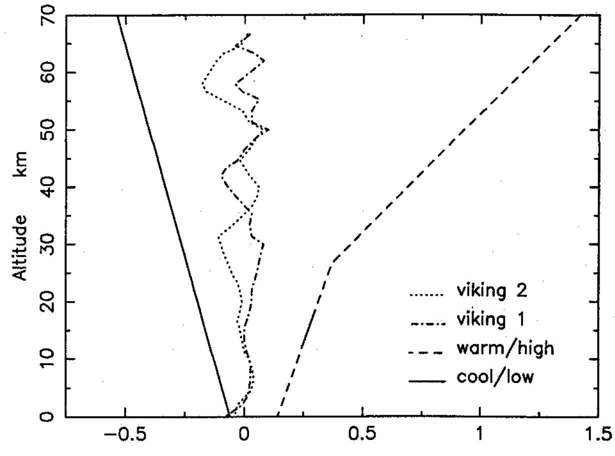

Fig. 7 Earth-scaled fractional density variations.

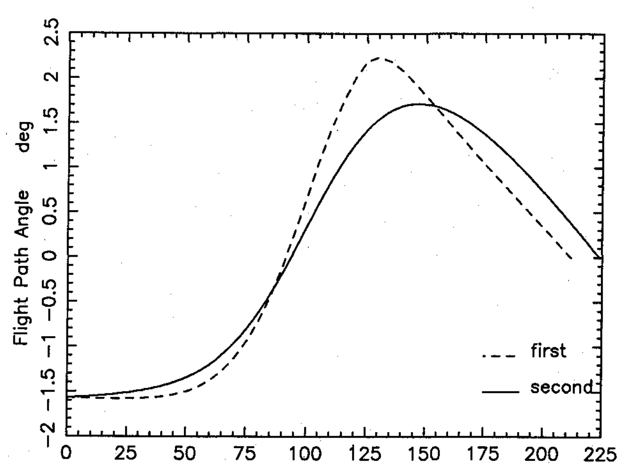

Fig. 6 Flight-path angle profile comparison.

Table 1 Atmospheric variation effects at exit

Atmosphere	r, km	θ, deg	ϕ, deg	v, km/s	ψ, deg	t, s
Nominal	6439.00	13.997	6.024	5.300	39.990	283
−50%	6439.00	15.925	7.282	5.297	39.995	326
+50%	6439.00	13.155	5.581	5.300	39.998	265
Viking 1	6439.00	13.978	5.986	5.297	40.019	282
Viking 2	6439.00	14.033	6.067	5.300	39.974	284
Cool/low pressure	6439.00	15.214	6.815	5.303	39.958	310
Warm/high pressure	6438.63	11.789	4.515	5.305	39.664	229

5.814 km/s. Use of four or five segments did not measurably improve the results. Reduction to a single segment, however, produced inferior results.

The second algorithm was run on the same data. Loh's function was obtained by sampling at an evenly spaced set of 25 points in $[0, U]$. The final Loh's term obtained through successive approximations appears in Fig. 3. The control histories appear in Fig. 4. Note that these are noticeably different than the controls generated by the previous algorithm. However, the resulting terminal speed, 5.819 km/s, is nearly identical. Raising or lowering the number of points in the Loh's term model by a factor of two did not alter the resulting terminal speed by more than 2 m/s.

Figures 5 and 6 depict the time histories of r and γ for both algorithms. These time histories can also be compared with the results of Hull, et al.,[4-6] Mishne and Speyer,[7] and Speyer and Crues.[8] It would appear that the results are similar to those achieved in Ref. 8 and superior to those of Refs. 4-6, although it is difficult to make a comparison since the results presented in Ref. 8 are for use of the algorithm as a guidance scheme with unmodeled error sources. The time histories of v and ψ do not differ significantly from one algorithm to the next.

As noted, the third algorithm was actually embedded online in a simulation as a guidance algorithm. At each guidance update, a new set of optimal controls was generated. As a test, a number of atmospheric density profiles were used and the variations in the exit conditions were examined. The nominal atmospheric profile was an exponential model with the parameters given earlier. Although the nominal atmosphere corresponds roughly to the Earth atmosphere, for atmospheric variations, we chose the relatively larger variations of Mars, and scaled them up to Earth density. The Martian variations were ±50% constant variation, the COSPAR warm/high pressure and cool/low pressure envelope limits, and the actual densities encountered by Viking 1 and 2 (see Ref. 15). Figure 7 contains plots of some of these scaled variations.

The desired exit conditions were

$$r_f = 6439.0 \text{ km}, \qquad v_f = 5.3 \text{ km/s}$$

$$\gamma_f = 0, \qquad \psi_f = 40.0 \text{ deg}$$

Guidance updates were performed every 10 s. A 25-point approximation to M was used, and the initial estimate of U was 8×10^3.

Some navigation assumptions were necessary in the simulation. Complete knowledge of the vehicle was assumed throughout. The current actual density was assumed known. The onboard model was fit to this current density simply by adjusting ρ_s; the scale height was not adjusted.

The results of these tests are tabulated in Table 1. The controlled exit states matched their desired values very well in the first six cases. For the last case, warm/high pressure, the results were not as good. In this last case, near the end of the trajectory, the navigation estimate of ρ_s steadily climbed. The navigation system did not adjust the scale height, which was actually in error. Thus, the controls always assumed a lower density profile than what would be subsequently encountered. Since a minimum angle-of-attack profile is roughly a minimum drag profile for a short trajectory, such as near the end in this case, the vehicle had insufficient control capability to deal with density levels that were consistently higher than expected. Note that had this higher-than-expected behavior occurred earlier (such as in the cool/low pressure case), the exit state deterioration would not have occurred. Clearly, this might also have been prevented by a navigation system that was able to detect the scale height error.

With regard to the uncontrolled exit states (downrange, crossrange, and time), the variation over all cases was approximately ±250 km and/or ±50 s. Of course, if the warm/high pressure case results improved, these variations would drop. How these variations compare with those of other guidance schemes has not yet been examined.

Discussion and Conclusions

Three optimal control algorithms applicable to aeromaneuvering guidance have been presented. The first two algorithms maximize terminal speed for a given plane change. The second algorithm appears to be an improvement over the first, as both an iteration on two unknowns and a piecewise linear fitting procedure were eliminated. For both of the first two algorithms, approximate optimality of the controls has been verified through comparison with the results of previous investigators. The third algorithm minimizes, in some sense, the control effort, whle attempting to control exit altitude, speed, flight-path angle, and heading angle. This third algorithm was implemented as a guidance scheme in a simulation. Results were generated that indicate that the algorithm performed quite well under rather severe unmodeled atmospheric density variations in all but one case. The relative failure in that one case might have been prevented through less naive navigation assumptions, although further testing with a full navigation filter would be required to support such a claim. All three algorithms are relatively simple, and could be implemented in an onboard guidance system.

An eventual goal is the development of guidance algorithms appropriate for the Mars Rover/Sample Return and Manned Mars Missions. Although these algorithms are producing quite good results, they are limited in two respects. First, making the assumption that Loh's term may be modeled as a function of the independent variable alone seems to require that the aerodynamic forces dominate. This does not pose a problem in the examples used here, since the atmospheric model is an approximation of the relatively dense Earth atmosphere. However, for the vehicle considered, this assumption is not supportable in a Martian environment, and the few trials run with a Martian atmospheric model have been less successful. A more accurate model, say including density dependence, might improve this. Also, the Hamilton-Jacobi-based technique of Mishne and Speyer[7] may, with sufficient terms, be able to handle this case, but no study has yet been performed. The second limitation is the number of states being controlled. As stated in the introduction, it would be ideal to be able to control all six exit states at a fixed exit time. The third guidance algorithm presented here controls only four exit states at an uncontrolled exit time. However, for the various atmospheric density models tested, the variation in the remaining states is not extreme, especially if the last case (warm/high pressure) can be improved.

Finally, one could note that the two controls generated by each of the last two algorithms have been constant and linear. For the first algorithm, they were nearly piecewise linear or constant. This would lead one to question whether a nonlinear programming code such as that employed by Bradt et al.,[1] with linear control segments, might not produce near optimal results. Such an algorithm is often computationally efficient, robust, and lends itself well to the addition of constraints.

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